

Robustness of scale invariance in models with self-organized criticality

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A random-neighbor extremal stick-slip model is introduced. In the thermodynamic limit, the distribution of states has a simple analytical form and the mean avalanche size, as a function of the coupling parameter, is exactly calculable. The system is critical only at a special point J_c in coupling parameter space. However, the critical region around this point, where approximate scale invariance holds, is very large, suggesting a mechanism for explaining the ubiquity of power laws in nature. [S1063-651X(99)06105-X]

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I. INTRODUCTION

Self-organized criticality (SOC) is an intriguing concept which started a large “avalanche” of research on mechanisms leading to scale invariance in extended dynamical systems [1]. However, there is no general agreement about ingredients necessary to create the self-organized critical state. This fact is reflected in the doubts about whether locally dissipative systems really present SOC or have only a very strong divergence of the mean avalanche size \bar{s} when approaching the conservative limit. The recent results by Chabanol and Hakim [2], Bröker and Grassberger [3], and Kinouchi *et al.* [4] stating that the random-neighbor Olami, Feder, and Christensen (OFC) model is not critical in the dissipative regime and contradicting previous claims [5] is a clear example of the difficulty of making such a distinction solely on the basis of simulations. It is also worth remembering that the prototypical sandpile (BTW) model is not critical in the presence of local dissipation [1,6,7].

The distinction between conservative and dissipative local dynamics, however, is not what is relevant for predicting critical behavior. The decisive ingredient seems to be the value of the coupling parameter J [or the nature of the distribution $p(J)$ in nonhomogeneous systems]. For example, the Feder and Feder model with k neighbors is nonconservative but is critical when the coupling constant is equal to $J_c = 1/k$ [3,8].

In this paper, a model is proposed which is similar to, but simpler than, the random-neighbor stick-slip models studied in [2,3]. For this model, the stationary distribution of states $p_\infty(E)$ and the mean avalanche size \bar{s} , as functions of the coupling parameter J , have simple analytical forms (in the limit of infinite system size). The analysis in terms of branching processes is transparent and gives a clear mechanism

for the emergence of very large but finite \bar{s} in a non-negligible region of the parameter space. In other words, although true criticality occurs only at a special point J_c , there exists a large region where power laws over several decades appear. In this region the behavior of the system can be considered almost critical.

This occurs because the original parameter which controls the critical behavior (the branching rate σ in a branching process) is now, in SOC models, a slow dynamical variable $\sigma_r(J)$ that depends on the coupling parameter J . In our model, the stationary value $\sigma_\infty(J)$ shows a plateau near the critical value $\sigma_c = 1$, thus enlarging the region in J space where the system displays a critical behavior. We will say that the system is critical for $J = J_c$ when $\sigma = 1$ and is “quasicalritical” or “almost critical” for values of J where $\sigma \sim 1$. This fact may be relevant as an explanation for the ubiquity of approximate scale invariance in nature [9].

The remainder of the paper is organized as follows: In Sec. II, the model is introduced and the main results obtained. The issue of robustness in SOC models is discussed in Sec. III. Section IV contains concluding remarks and suggestions for future work.

II. EXTREMAL FEDER AND FEDER MODEL (EFF MODEL)

A. The model

The EFF model is a random-neighbor version of the Feder and Feder model [3,8] using an extremal dynamics similar to the Bak-Sneppen model [10]. The extremal dynamics, which in this case substitutes for (and plays the same role as) the slow driving of the original Feder and Feder model, is here an essential ingredient for the observation of self-organized criticality.

All sites $j = 1, \dots, N$ have a continuous state variable $E_j \in \mathcal{R}$. At each time step the site with maximal value “fires,” resetting its value to zero plus a noise term η . Then, k random “neighbors” (rn) of the firing site have their

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states incremented by a constant J plus a noise term. The choice of neighbors is done at the firing instant: the randomness is *annealed*. So, denoting the extremal value at instant t as $E_i^* \equiv \max\{E_j\}$, the update rules are

$$\begin{aligned} E_i^*(t+1) &= \eta(t), \\ E_{\text{RN}}(t+1) &= E_{\text{RN}}(t) + J + \eta_{\text{RN}}(t), \end{aligned} \quad (1)$$

with η and η_{RN} being random variables uniformly distributed in the interval $[0, \epsilon]$ (the range of ϵ will be discussed later). Note that each random neighbor receives a different quantity η_{RN} .

Consider the instantaneous density of states $p_t(E)$. It is clear that for any E outside the intervals $I_n \equiv [(n-1)J, (n-1)J + n\epsilon]$, $n=1, 2, \dots$, this density decays to zero for long times. These intervals effectively discretize the phase space, so it is useful to define the following quantities:

$$P_n = \int_{(n-1)J}^{(n-1)J + n\epsilon} p(E) dE, \quad (2)$$

with $n=1, 2, \dots, n_{\text{max}}$, and $\epsilon < J/n_{\text{max}}$ so that the intervals do not overlap (the integer n_{max} will be obtained later). The process can be thought of as a transference of sites between the intervals I_n . At each time step, one site is transferred to the interval I_1 and, with probability kP_1 , one site is removed from this interval. The average flux to the intervals I_n with $n > 1$ corresponds to the probability kP_{n-1} that a neighbor is chosen in the previous I_{n-1} interval minus the probability kP_n that a neighbor is chosen in the interval I_n . The average number of sites in each interval is $N_n(t) = NP_n(t)$. For long times, that is, when the density of states outside the I_n intervals goes to zero, one can write

$$P_1(t+1) = P_1(t) + \frac{1}{N} [1 - kP_1(t)], \quad (3)$$

$$P_n(t+1) = P_n(t) + \frac{1}{N} [kP_{n-1}(t) - kP_n(t)].$$

Here, each time step is equal to the update of the maximal site and k random neighbors.

The condition for steady states, $P_n(t+1) = P_n(t) = P_n^*$, gives

$$\begin{aligned} P_1^* &= 1/k, \\ P_n^* &= P_{n-1}^*, \end{aligned} \quad (4)$$

that is, $P_n^* = 1/k$ for all n . But since $p(E)$ is normalized, only n_{max} intervals with P_n of $O(1)$ can exist. That is,

$$\sum_{n=1}^{n_{\text{max}}} P_n^* = n_{\text{max}} \times \frac{1}{k} = 1, \quad (5)$$

giving that $n_{\text{max}} = k$. This means that $p_\infty(E)$ is composed of k bumps ($n=1, \dots, n_{\text{max}}=k$) and the previous condition for producing nonoverlapping intervals I_n reads $\epsilon < J/k$. There is also a bump of $O(\ln N/N)$ (by analogy with the results from

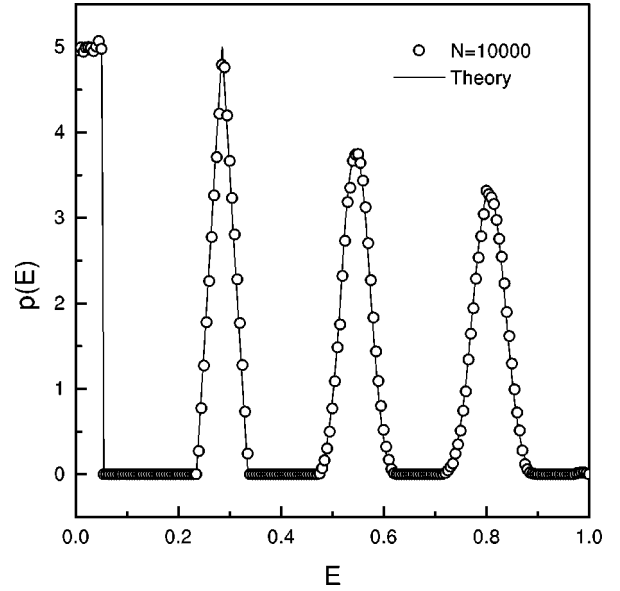


FIG. 1. Distribution of states $p_\infty(E)$ for $k=4$, $J=0.235$, and $\epsilon=0.05$: theoretical (solid) and simulation (circles) with $N=10^4$ sites.

[11]) situated at the interval $I_{k+1} = [kJ, kJ + (k+1)\epsilon]$. The other intervals $n > k+1$ have P_n of yet smaller order (see Fig. 1).

B. Avalanches

An avalanche will be defined as the number of firing sites until an extremal site value falls below the threshold $E_{\text{th}} = 1$ [13]. Note that the first site of an avalanche (the ‘‘seed’’) always has $E < 1$ but it counts as a firing site. So, if a seed produces no suprathreshold sites (‘‘descendants’’), this counts as an avalanche of size one. This definition of avalanches agrees with that used in the studies of relaxation oscillator models.

In these random-neighbor models, an avalanche can be identified as a branching process where an active site produces k new sites, each one having a probability p of being active (a ‘‘branch’’) and a probability $1-p$ of being inactive (a ‘‘leaf’’). The branching rate $\sigma = kp$ measures the probability that a firing site produces another firing site.

A known result for a process with a constant branching rate σ is the distribution of avalanche sizes [3],

$$P(s) = \frac{1}{s} \binom{ks}{s-1} \left(1 - \frac{\sigma}{k}\right)^{ks - (s-1)} \left(\frac{\sigma}{k}\right)^{s-1}, \quad (6)$$

which, for large s and small $\delta = 1 - \sigma$, has the form

$$P(s) \approx \frac{1}{\sqrt{2\pi(1-1/k)}} s^{-3/2} \exp(-s/s_\xi), \quad (7)$$

$$s_\xi \approx \frac{2(k-1)}{k} (1-\sigma)^{-2}. \quad (8)$$

We will see that Eq. (6) can be applied to the EFF model with the stationary value $\sigma_\infty(J)$.

Now, consider an avalanche which has terminated after s sites have fired. This avalanche is composed of one seed and $s-1$ descendants. But the average number of descendants produced by s firing sites is σs . Thus, on average, the relation

$$\bar{s} - 1 = \sigma \bar{s} \quad (9)$$

must hold, which leads to

$$\bar{s} = \frac{1}{1 - \sigma}. \quad (10)$$

Of course, this result can be obtained directly from Eq. (6) after some work. Note that $\sigma_\infty \equiv \sigma(t \rightarrow \infty)$ refers to the stationary value of the branching rate: during the transient, σ_t changes with the avalanche time t . Although questioned by some authors [6], we retain the name *self-organization* for this evolution of σ_t toward σ_∞ mainly as a label to distinguish these systems from standard branching processes where σ is fixed *a priori*.

C. The $J=1/k$ case

In the case $J=1/k$, the calculation of σ_∞ is trivial. The k th bump ($n=n_{\max}$), which starts at $(k-1)J$, must lie below the threshold $E_{\text{th}}=1$ (if not, the system is supercritical). Then, ϵ must satisfy the condition $(k-1)/k + k\epsilon < 1$, that is,

$$\epsilon < 1/k^2. \quad (11)$$

For the standard $k=4$ neighbor case this reads $\epsilon < 0.0625$. This condition also implies that neighbors pertaining to the other bumps do not contribute to σ_∞ , that is, cannot fire when receiving a maximal contribution $J + \epsilon$. Now, since all the neighbors pertaining to the k th bump receive at least the quantity $J=1/k$, they are always transformed into active sites. Thus, the average number of descendants of a firing site is

$$\sigma_\infty = kP_k^* = k \times \frac{1}{k} = 1, \quad (12)$$

which corresponds to a critical branching process. It is known that in this case the system presents an infinite \bar{s} [see Eq. (10)] and, for large s , a pure power law

$$P(s) = \frac{1}{\sqrt{2\pi(1-1/k)}} s^{-3/2} \quad (13)$$

for the distribution of avalanche sizes [3].

D. Results for general J

For the case $J < 1/k$, in order to obtain an expression for $\sigma_\infty(J)$, the knowledge of the distribution of states $p_\infty(E)$ is required. But it is clear that if $kJ=1-\delta$ then inevitably $\sigma_\infty < 1$ (even for very small $\delta > 0$), since some sites pertaining to the k th bump may not receive a sufficient contribution to make them active [see Eq. (17) below]. Thus, any value $J < J_c = 1/k$ is subcritical. This is a common feature of many models with SOC [3,6,14].

In our model, the calculation of $p_\infty(E)$ is very simple. In the stationary state, a site pertaining to the n th bump has energy $E=(n-1)J+z_n$, where z_n is the sum of n random variables uniformly distributed in the interval $[0, \epsilon]$. The distribution $p(z_n)$ may be calculated from

$$p(z_1) = \epsilon^{-1} \Theta(z_1) \Theta(\epsilon - z_1),$$

$$p(z_{n+1}) = \int_{-\infty}^{\infty} dz_n dz_1 p(z_n) p(z_1) \delta(z_n + z_1 - z_{n+1}).$$

For the $k=4$ case,

$$p(z_2) = \epsilon^{-2} z_2 \Theta(z_2) \Theta(\epsilon - z_2) + (2\epsilon - z_2) \Theta(z_2 - \epsilon) \Theta(2\epsilon - z_2), \quad (14)$$

$$p(z_3) = \epsilon^{-3} \frac{z_3^2}{2} \Theta(z_3) \Theta(\epsilon - z_3) + \left(-z_3^2 + 3\epsilon z_3 - \frac{3\epsilon^2}{2} \right) \Theta(z_3 - \epsilon) \Theta(2\epsilon - z_3) + \left(\frac{z_3^2}{2} - 3\epsilon z_3 + \frac{9\epsilon^2}{2} \right) \Theta(z_3 - 2\epsilon) \Theta(3\epsilon - z_3),$$

$$p(z_4) = \epsilon^{-4} \frac{z_4^3}{6} \Theta(z_4) \Theta(\epsilon - z_4) + \left(-\frac{z_4^3}{2} + 2\epsilon z_4^2 - 2\epsilon^2 z_4 + \frac{2\epsilon^3}{3} \right) \times \Theta(z_4 - \epsilon) \Theta(2\epsilon - z_4) + \left(-\frac{x^3}{3} + 2\epsilon x^2 - 2\epsilon^2 x + \frac{2\epsilon^3}{3} \right) \times \Theta(z_4 - 2\epsilon) \Theta(3\epsilon - z_4) + \frac{x^3}{6} \Theta(z_4 - 3\epsilon) \Theta(4\epsilon - z_4), \quad (15)$$

with the shorthand $x \equiv (4\epsilon - z_4)$. The distribution $p_\infty(E)$ has k bumps. Each bump (labeled by n) starts at $E_n = (n-1)J$, being proportional to $p(z_n)$ (the constant of proportionality is just $1/k$). In Fig. 1, the distribution $p_\infty(E)$ is compared with simulation results for a system with 10^4 sites, $J=0.235$, $\epsilon=0.05$, and a sufficient number of avalanches.

For such large systems, we must be careful about using reliable random-neighbor generators. In order to speed up the search for the extremal site, we used the binary rooted tree algorithm described by Grassberger [12]. For example, if the system has 2^m sites, a binary tree with $m+1$ levels is created such that, in each node at level l , is stored the largest value of E of the two branch nodes of the $(l+1)$ th level. So, the 0th (root) level contains the value of the extremal site. Ascending the tree, we locate the position of this site in the upper level. After the extremal site firing, the tree must be updated. The same occurs when the random neighbors are updated. These operations have a complexity $O(\ln \mathcal{N})$ instead of the $O(\mathcal{N})$ complexity of the naive search mechanism.

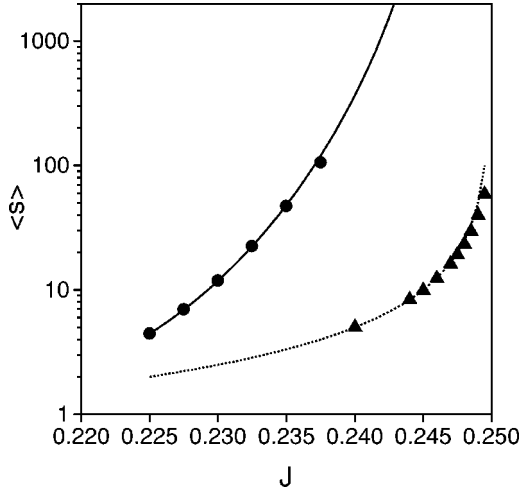


FIG. 2. Mean avalanche size \bar{s} as a function of parameter J . Theoretical (solid) and simulations with N up to $2^{18}=262\,144$ sites for noisy EFF model (circles) with $\epsilon=0.05$ and noiseless EFF model (triangles) with $\epsilon=0.2$. These ϵ values are chosen such that the last interval (I_4) has the same length in both models.

The stationary branching rate σ_∞ is calculated as follows. All the sites that can be activated pertain to the k th bump. When hit, sites with $E > 1 - J$ are always activated. In terms of the rescaled variable $z_k = E - (k-1)J$, this condition refers to sites with $z_k > \delta \equiv 1 - kJ$. They contribute to the branching rate with the quantity σ' ,

$$\sigma' \equiv k \int_{1-J}^1 p(E) dE = \int_{\delta}^{\delta+J} p(z) dz, \quad (16)$$

where $z \equiv z_k$.

Sites with $E < 1 - J - \epsilon$ cannot be activated and do not contribute to σ . Sites with $1 - J - \epsilon < E < 1 - J$ can be activated if they receive a quantity $J + \eta > 1 - E$, that is, $\eta > \delta - z$. This occurs with probability $P(\eta > \delta - z) = 1 - (\delta - z)/\epsilon$. Thus, these sites contribute to the branching rate with the quantity

$$\begin{aligned} \sigma'' &\equiv k \int_{1-kJ-\epsilon}^{1kJ} P(E) P(\eta > 1 - E - J) dE \\ &= \int_{\delta-\epsilon}^{\delta} p(z) \left(1 - \frac{\delta-z}{\epsilon}\right) dz. \end{aligned} \quad (17)$$

The total branching rate is then

$$\begin{aligned} \sigma_\infty &= \sigma' + \sigma'' \\ &= 1 - \int_0^{\delta-\epsilon} p(z) dz - \frac{\delta}{\epsilon} \int_{\delta-\epsilon}^{\delta} p(z) dz + \frac{1}{\epsilon} \int_{\delta-\epsilon}^{\delta} z p(z) dz, \end{aligned} \quad (18)$$

where we used the fact that $\int_0^{\delta+J} p(z) dz = 1$. Since $p(z)$ has a simple piecewise polynomial form [see Eq. (14)] the calculation of \bar{s} is straightforward and the result is presented in Fig. 2 along with simulation results for the $k=4$, $\epsilon=0.05$, for systems with up to $N=2^{18}=262\,144$ sites. In Fig. 3, we plot simulation results for the $P(s)$ distribution which agree

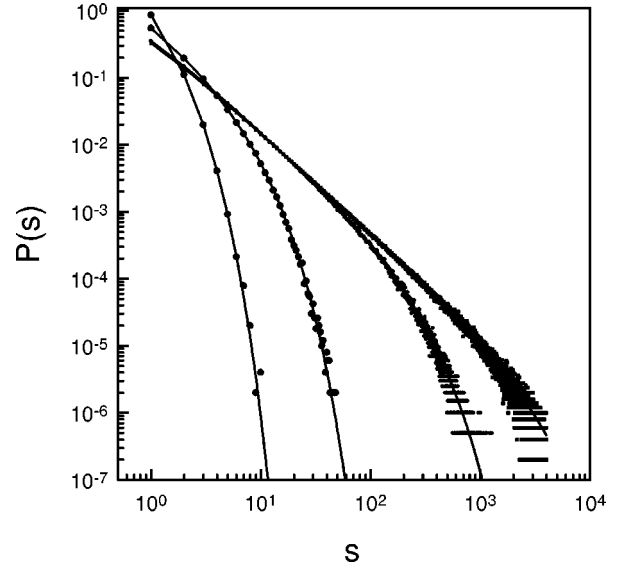


FIG. 3. Simulation results ($N=2^{13}=8192$ sites, $k=4$, $\epsilon=0.05$) for the distribution $P(s)$ with $J=0.21, 0.22, 0.23, 0.235$ (from left to right), compared with theoretical curves (solid).

very well with Eq. (6) if $\sigma = \sigma_\infty(J)$ is used in that expression. Strong finite size effects, however, are present when $J > 0.235$.

For $\delta < \epsilon$, that is, $J_c - J < \epsilon/k$, the form assumed by σ_∞ is particularly simple, since $p(z) = C \epsilon^{-k} z^{k-1}$ in that interval (C is a numerical constant). Then,

$$\sigma_\infty = 1 - \frac{C}{\epsilon^{k+1}} \int_0^{\delta} z^{k-1} (\delta - z) dz = 1 - \frac{C}{k(k+1)} \left(\frac{\delta}{\epsilon}\right)^{k+1}, \quad (19)$$

the avalanche cutoff length.

Since $\delta \equiv 1 - kJ = k(J_c - J)$, we obtain, from Eqs. (8) and (10), the avalanche cutoff size and the average avalanche size

$$\begin{aligned} s_\xi &= \frac{2(k-1)(k+1)^2 \epsilon^{2(k+1)}}{C^2 k^{2k+1}} (J_c - J)^{-\nu}, \\ \bar{s} &= \frac{(k+1) \epsilon^{k+1}}{C k^k} (J_c - J)^{-\nu/2}, \end{aligned} \quad (20)$$

with the critical exponent

$$\nu = 2(k+1). \quad (21)$$

For example, with $k=4$ [which means $C=1/6$, see Eq. (14)] and $\epsilon=0.05$, the mean avalanche size is $\bar{s}=120$ already for $J=0.2375$. Curiously, this behavior is similar to the $\bar{s} \propto (J_c - J)^{-k}$ divergence found in the standard random-neighbor FF model [3].

E. The EFF model with noiseless couplings

It is instructive to compare the above behavior with that of a simpler EFF model [4] where the firing rule is the same, $E_i^*(t+1) = \eta \in [0, \epsilon]$, but the coupling between sites is

noiseless, $E_{rn}(t+1) = E_{rn}(t) + J$. Thus, $p_\infty(E)$ assumes the form of k rectangular bumps with $p(z_n) = \epsilon^{-1} \Theta(z_n) \Theta(\epsilon - z_n)$. In this *noiseless* EFF model, the branching rate, the cutoff size, and the average avalanche size are

$$\sigma_\infty = \begin{cases} 0 & \text{for } \delta > \epsilon \\ 1 - \delta/\epsilon & \text{for } 0 < \delta < \epsilon, \end{cases}$$

$$s_\xi = 2(k-1) \frac{\epsilon^2}{k^3} (J_c - J)^{-2}, \quad (22)$$

$$\bar{s} = \frac{\epsilon}{k} (J_c - J)^{-1}.$$

In contrast with the noisy model, large avalanches only occur when J is very close to J_c (see Fig. 2). Thus, the EFF model with noiseless couplings does not present an enlargement of the region where the system displays a critical behavior as observed in the noisy EFF model.

III. ON SOC DEFINITIONS

The idea of self-organized criticality present in the literature embodies two distinct properties. The term *critical* refers to the existence of power laws and to the absence of a characteristic scale in the response of the system to the driving mechanism of the dynamics; the term *self-organized* refers to the fact that there exists a parameter (σ_t), which controls the avalanching process, whose value is not fixed *a priori* as, for example, in standard percolation and branching processes. This parameter evolves in time, during a transient phase, toward a stationary value σ_∞ . Indeed, this time dependence should be written as $\sigma_t = \sigma(p_t(E))$, that is, σ_t is a functional of the distribution of states $p_t(E)$, that, in turn, evolves toward a statistically stationary distribution $p_\infty(E)$. So, $\sigma_\infty \equiv \sigma(p_\infty(E))$. If $\sigma_\infty = \sigma_c = 1$, the system is critical.

The evolution of $p_t(E)$ toward the steady state $p_\infty(E)$ is akin to the transient relaxation in equilibrium systems: any initial condition leads to the same stationary state, thus to the same value of σ_∞ . However, this robustness to initial conditions and external perturbations on $p(E)$ (“dynamical stability”) should not be mistaken as parameter robustness (“structural stability”). This is a distinct characteristic claimed to be present on some SOC models (see, for instance, [1,15,16,18]). For a system to have “structural stable criticality,” there would be a finite parameter range for which, after the transient, the system is critical. In this case, $\sigma_\infty(J) = \sigma_c$ for J belonging to some interval $[J_c, 1/k]$. We will call that kind of behavior *generic SOC*.

“Structural stability” is a relative concept which depends on the parameter space physically available for the system. For example, it is well known that the sandpile model is not critical in the presence of dissipation. The sandpile dissipation parameter corresponds to the quantity $\delta = 1 - kJ$ in our model [6,7]. The standard BTW model is by definition “tuned” into a critical state through the “imposition” of a conservation law. Although it could be argued that dissipation is not a natural feature of sandpiles, since sand does not disappear, the appearance of SOC in nature would sound much more natural if criticality could be observed over a

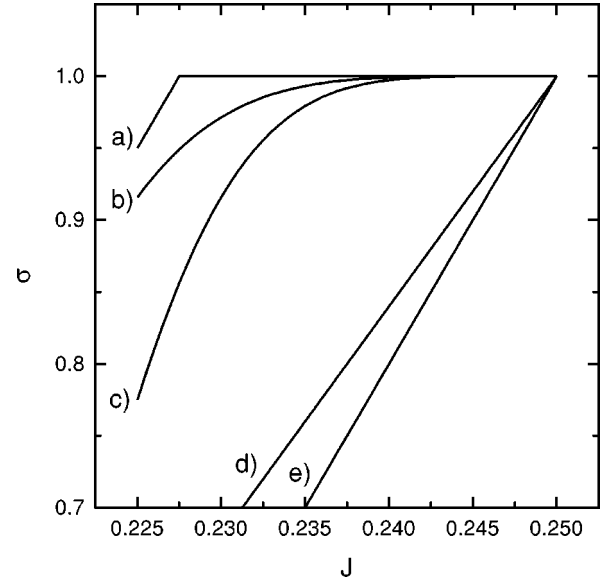


FIG. 4. (a) Generic self-organized criticality: the value of parameter σ_∞ is critical on a finite range of the system parameter J ; (b) $\epsilon = 0.0625$ and (c) $\epsilon = 0.05$, enlargement of the critical region (EFF model with noisy couplings, $k=4$): σ_∞ is almost constant near J_c ; (d) $\epsilon = 0.25 = 4 \times 0.0625$ and (e) $\epsilon = 0.2 = 4 \times 0.05$, standard critical behavior (EFF model with noiseless couplings): the coupling parameter J must be very close to 0.25 for obtaining $\sigma_\infty \approx \sigma_c$ due to the linear behavior of $\sigma_\infty(J)$. Note that, in the noiseless couplings case, ϵ refers to the amplitude of the noise received by the extremal site after discharge.

region of the parameter space, not only in a special point.

Generic self-organized criticality is depicted in curve (a) of Fig. 4. In this case, there is a finite range of J values for which σ_∞ assumes the critical value $\sigma_c = 1$. In this figure, curves (d) and (e) represent the behavior observed in the BTW model and also in the noiseless EFF model examined above, for which the system is critical only for a special value of the parameter J . However, there is a third possibility. Curves (b) and (c) represent the behavior of $\sigma(J)$ given by Eq. (18) for the EFF model with noisy couplings: although the system is critical only at $J = J_c$, the system is “almost critical” over a large parameter region. This behavior has also been observed in the standard random-neighbor versions of FF and OFC models [3]. The importance of this characterization is that several models in the SOC literature, previously seen as having true generic criticality, are now recognized as having only an almost critical behavior as discussed above.

A model which apparently presents generic SOC behavior in coupling space is the two-dimensional (2D) OFC model [16,17,19]. Also the standard Feder and Feder model [8] is claimed to be critical for $J < J_c$ [18,19]. Looking at the behavior of the models studied so far, we make the following conjecture: a necessary condition for a lattice model to present a generic SOC behavior is that its corresponding random-neighbor version already presents an enlarged critical region in the sense discussed above. This could be tested by comparing the 2D versions of the EFF and noiseless EFF models studied above.

In conclusion, we found that some systems that display SOC, although being critical only for a single value for J , are

almost critical in a large region of the parameter space. This almost critical behavior is difficult to distinguish, in practice, from true generic SOC behavior: both in numerical simulations (huge lattices would have to be used) and in nature (due to limitations in the data) power laws can only be measured over some scale decades [9]. So, in order to explain the ubiquity of scale invariance in nature, having a true generic SOC or only presenting an enlarged region where the system is almost critical are, as far as one can measure, identical.

IV. CONCLUSIONS

A class of extremal stick-slip models has been introduced and studied in the $N \rightarrow \infty$ limit. We showed that noise in the couplings of the EFF model changes the exponent that controls the amplitude of the critical region from $\nu=2$ to $\nu = 2(k+1)$. This enlargement of the region where the system displays a critical behavior is similar to that found in the standard random-neighbor OFC and FF models [2–4]. As in other models, the true critical state occurs only for one point in parameter space [2,3,6,7,14], but in practice that fact can

hardly be noticed, and the model displays the typical features of generic SOC.

In future work we hope to determine the minimal ingredients for producing the enlargement of the critical region in the models examined in the SOC literature. We will also present results for the two-dimensional case and compare with the standard OFC and FF models. The simple mechanism devised in this work suggests that, if true generic criticality is not easy to obtain in the space of possible models, this quasicritical behavior certainly is. Thus, for explaining the robustness of approximate scale invariance in nature, this mechanism seems to be more “generic” than generic criticality.

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